

ON THE STRUCTURE OF LIPSCHITZ-FREE SPACES

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ABSTRACT. In this note we study the structure of Lipschitz-free Banach spaces. We show that every Lipschitz-free Banach space over an infinite metric space contains a complemented copy of ℓ_1 . This result has many consequences for the structure of Lipschitz-free Banach spaces. Moreover, we give an example of a countable compact metric space K such that $\mathcal{F}(K)$ is not isomorphic to a subspace of L_1 and we show that whenever M is a subset of \mathbb{R}^n , then $\mathcal{F}(M)$ is weakly sequentially complete; in particular, c_0 does not embed into $\mathcal{F}(M)$.

INTRODUCTION

Given a metric space M , it is possible to construct a Banach space $\mathcal{F}(M)$ in such a way that the Lipschitz structure of M corresponds to the linear structure of $\mathcal{F}(M)$. This space $\mathcal{F}(M)$ is sometimes called “Lipschitz-free space”. We refer to the next section for some more details concerning the construction and basic properties of those spaces. Although Lipschitz-free spaces over separable metric spaces are easy to define, their structure is poorly understood to this day. The study of the linear structure of Lipschitz-free spaces over metric spaces has become an active field of study, see e.g. [8, 9, 11, 13, 14, 18, 20]. In the first part of this paper we prove the following general result.

Theorem 1. *Let M be an infinite metric space. For the Banach space $X = \mathcal{F}(M)$, we have*

- (i) $\ell_1 \xhookrightarrow{c} X$, i.e., there is a complemented subspace of X isomorphic to ℓ_1 .

From this we get

- (ii) $X \not\hookrightarrow \mathcal{C}(K)$, i.e., X is not isomorphic to a complemented subspace of a $\mathcal{C}(K)$ space.
- (iii) X^* is not weakly sequentially complete; in particular, X is not isomorphic to L^1 -predual.
- (iv) X is not isomorphic to the Gurarii space.
- (v) X is a projectively universal separable Banach space, i.e., for any separable Banach space Y there exists a bounded linear operator from X onto Y .

It often happens that the Lipschitz-free space over a “small enough” space is isomorphic to ℓ_1 . For example, if $M \subset \mathbb{R}$ is a set of measure zero or if M is a separable ultrametric space, then $\mathcal{F}(M)$ is isomorphic to ℓ_1 , see [13] and [7]. By the result of A. Dalet [8], $\mathcal{F}(K)$ is a dual space with MAP whenever K is a countable compact metric space. Hence, one could conjecture that in this case $\mathcal{F}(K)$ is isomorphic to ℓ_1 . We give an example which shows that this is not the case.

Theorem 2. *There is a countable compact metric space K such that $\mathcal{F}(K) \not\hookrightarrow L_1$, i.e., $\mathcal{F}(K)$ is not linearly isomorphic to a subspace of L_1 . Moreover, K is a convergent sequence, i.e., it has only one accumulation point.*

If M contains a bi-Lipschitz copy of c_0 , then $\mathcal{F}(M)$ is an isomorphically universal separable Banach space; i.e., $\mathcal{F}(M)$ contains an isomorphic copy of every separable Banach space (for more details we refer to Section 4). Y. Dutrioux and V. Ferenczi in [9] asked for the converse. The answer to this question is in general negative, because it follows from the result of P. Kaufmann

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[18, Corollary 3.3] that $\mathcal{F}(c_0)$ is isomorphic to $\mathcal{F}(B_{c_0})$ (thus, it is a universal) and of course, since B_{c_0} is bounded, B_{c_0} does not contain a bi-Lipschitz copy of c_0 . However, we can still ask the question in the setting of Banach spaces.

Question 1. *Let X be a Banach space. Is $\mathcal{F}(X)$ universal if and only if X contains a bi-Lipschitz copy of c_0 ?*

The following result is a partial progress towards the answer to this question. Up to our knowledge, Question 1 is left open.

Theorem 3. *Let $M \subset \mathbb{R}^n$ be an arbitrary set. Then $\mathcal{F}(M)$ is weakly sequentially complete. Consequently, $c_0 \not\hookrightarrow \mathcal{F}(M)$; i.e., c_0 is not linearly isomorphic to a subspace of $\mathcal{F}(M)$.*

To the best of our knowledge, it was not even known whether there could be a Lipschitz-free space which neither embeds into L_1 nor is universal. The example given in Theorem 2 is one such example (because, by the result of A. Dalet [8], $\mathcal{F}(K)$ is a separable dual space and so it does not contain c_0). Another one is the space $\mathcal{F}([0, 1]^n)$, see Theorems 3 and 7.

In the last section of this note we mention some open problems related to the structure of isomorphically universal Lipschitz-free Banach spaces.

The notation and terminology we use are relatively standard. If X and Y are Banach spaces, the symbol $Y \hookrightarrow X$ (resp. $Y \not\hookrightarrow X$) means that Y is (resp. is not) linearly isomorphic to a subspace of X . If (M, d) is a metric space, $x \in M$ and $r \geq 0$, we use $U(x, r)$ and $B(x, r)$ to denote respectively the open and closed ball, i.e., the set $\{y \in M : d(x, y) < r\}$ and $\{y \in M : d(x, y) \leq r\}$.

1. BASIC FACTS ABOUT LIPSCHITZ-FREE SPACES

Let (M, d) be a metric space with a distinguished point denoted by 0. Consider the space $\text{Lip}_0(M)$ of all real-valued Lipschitz functions that map $0 \in M$ to $0 \in \mathbb{R}$. It has a vector space structure and one can define a norm $\|\cdot\|_{\text{Lip}}$ on $\text{Lip}_0(M)$, where for $f \in \text{Lip}_0(M)$, $\|f\|_{\text{Lip}}$ is the minimal Lipschitz constant, i.e., $\sup\{\frac{|f(x)-f(y)|}{d(x,y)} : x \neq y \in M\}$. Then $(\text{Lip}_0(M), \|\cdot\|_{\text{Lip}})$ is a Banach space.

For any $x \in M$ denote by $\delta_x \in \text{Lip}_0(M)^*$ the evaluation functional, i.e., $\delta_x(f) = f(x)$ for every $f \in \text{Lip}_0(M)$. Denote by $\mathcal{F}(M)$ the closure of the linear span of $\{\delta_x : x \in M\}$ with the dual space norm denoted simply by $\|\cdot\|$. Observe that for any $x, y \in M$ we have $\|\delta_x - \delta_y\| = d(x, y)$.

This space is usually called Lipschitz-free Banach space (also Arens-Eells space) and it is uniquely characterized by the following universal property.

Let X be a Banach space and suppose $L : M \rightarrow X$ is a Lipschitz map such that $L(0) = 0$. Then there exists a unique linear map $\widehat{L} : \mathcal{F}(M) \rightarrow X$ extending L , i.e., the following diagram commutes

$$\begin{array}{ccc} M & \xrightarrow{L} & X \\ \delta_M \downarrow & & \downarrow \text{id}_X \\ \mathcal{F}(M) & \xrightarrow{\widehat{L}} & X \end{array}$$

and $\|\widehat{L}\| = \|L\|_{\text{Lip}}$ where $\|\cdot\|_{\text{Lip}}$ denotes the Lipschitz norm of L .

This fact is usually referred to as folklore. The proof is so simple that we include it here.

Fix a Banach space X and a Lipschitz map $L : M \rightarrow X$ mapping 0 to 0. Extend linearly L from M onto $\text{span}\{\delta_x : x \in M\}$ and denote this extension by \widehat{L} . We only need to check that $\|\widehat{L}\|_{\text{Lip}} = \|L\|_{\text{Lip}}$. Pick some $a \in \text{span}\{\delta_x : x \in M\}$. Then $\|\widehat{L}(a)\|_X = f(\widehat{L}(a))$ for some

$f \in B_{X^*}$. However, $f \circ L$ then belongs to $\text{Lip}_0(M)$ and $\|f \circ L\|_{\text{Lip}} \leq \|L\|_{\text{Lip}}$. It follows that $\|a\| \|L\|_{\text{Lip}} \geq \|\widehat{L}(a)\|_X$ which proves the claim. Then we can extend \widehat{L} to $\mathcal{F}(M)$, the closure of $\text{span}\{\delta_x : x \in M\}$.

Using this universal property of $\mathcal{F}(M)$, it is immediate that $\mathcal{F}(M)^* = \text{Lip}_0(M)$. Indeed, it is enough to consider $X = \mathbb{R}$ in the universal property mentioned above.

Further, it is useful to observe that whenever N is a subspace of a metric space (M, d) , then $\mathcal{F}(N)$ is linearly isometric to a subspace of $\mathcal{F}(M)$. Indeed, the isometry is determined by sending $\delta_x \in \mathcal{F}(N)$ to $\delta_x \in \mathcal{F}(M)$; in order to see it is an isometry it is enough to use the well-known fact that any $f \in \text{Lip}_0(N)$ can be extended to $F \in \text{Lip}_0(M)$ with $\|f\|_{\text{Lip}} = \|F\|_{\text{Lip}}$, e.g., by putting $F(x) := \inf\{f(n) + \|f\|_{\text{Lip}} d(n, x) : n \in N\}$, $x \in M$; see e.g. [16, Lemma 7.39]. Using this observation together with the universal property of $\mathcal{F}(M)$ we see that the Lipschitz structure of M corresponds to the linear structure of $\mathcal{F}(M)$. For example, if N is bi-Lipschitz equivalent (resp. isometric) to a subset of M , then $\mathcal{F}(N)$ is linearly isomorphic (resp. linearly isometric) to a subspace of $\mathcal{F}(M)$, etc.

The last basic fact we would like to mention here is that it is possible to give an ‘internal’ definition of the norm on $\mathcal{F}(M)$, i.e. by a formula which refers only to the metric on the metric space M . This is in contrast to the ‘external’ definition given above which refers to the space $\text{Lip}_0(M)$ in the computation of the norm. This is described e.g. in [27]. The proof is not difficult and so we include it here as well.

Let us consider another norm, denoted by $\|\cdot\|_{KR}$, on $\text{span}\{\delta_x : x \in M\}$ which is a variant of the so-called Kantorovich-Rubinstein metric, a concept that penetrated many areas of mathematics and computer science. Let us identify δ_0 with $0 \in \mathcal{F}(M)$. For $a \in \text{span}\{\delta_x : x \in M \setminus \{0\}\}$ set

$$\|a\|_{KR} = \inf\{|\alpha_1| \cdot d(y_1, z_1) + \dots + |\alpha_n| \cdot d(y_n, z_n) : a = \alpha_1(\delta_{y_1} - \delta_{z_1}) + \dots + \alpha_n(\delta_{y_n} - \delta_{z_n})\}.$$

It is straightforward to check that $\|\cdot\|_{KR}$ is a seminorm. Moreover, it is the largest seminorm $\|\cdot\|'$ on $\text{span}\{\delta_x : x \in M\}$ satisfying $\|\delta_x - \delta_y\|' \leq d(x, y)$ for every $x, y \in M$. Indeed, any seminorm $\|\cdot\|'$ with that property must satisfy the inequality $\|x\|' \leq |\alpha_1| \|\delta_{y_1} - \delta_{z_1}\|' + \dots + |\alpha_n| \|\delta_{y_n} - \delta_{z_n}\|'$ when $x = \alpha_1(\delta_{y_1} - \delta_{z_1}) + \dots + \alpha_n(\delta_{y_n} - \delta_{z_n})$ which shows that $\|x\|' \leq \|x\|_{KR}$. Since the standard norm $\|\cdot\|$ on $\mathcal{F}(M)$ satisfies the condition, we get that $\|\cdot\| \leq \|\cdot\|_{KR}$ which implies that $\|\cdot\|_{KR}$ is actually a norm and that $\|\delta_x - \delta_y\|_{KR} = d(x, y)$ for every $x, y \in M$.

Consider now the identity mapping $L : M \rightarrow \overline{\text{span}\{\delta_x : x \in M\}}^{\|\cdot\|_{KR}}$ sending x to δ_x . It is an isometric embedding. By the universality property of $\mathcal{F}(M)$, L extends to $\widehat{L} : \mathcal{F}(M) \rightarrow \overline{\text{span}\{\delta_x : x \in M\}}^{\|\cdot\|_{KR}}$ which is still 1-Lipschitz. It follows that $\|\cdot\|_{KR} \leq \|\cdot\|$, so the norms $\|\cdot\|$ and $\|\cdot\|_{KR}$ are one and the same. This fact is often referred to as the Kantorovich duality.

2. EMBEDDING OF ℓ_1

The purpose of this section is to prove Theorem 1. It will be deduced from the fact that ℓ_∞ embeds in the dual of a Lipschitz-free space, i.e., into the space of Lipschitz functions. Let us note that we do not know whether ℓ_∞ embeds isometrically into $\text{Lip}_0(M)$ for every infinite metric space M . The natural way of embedding ℓ_∞ into the space of Lipschitz functions is described in the Lemma below.

Lemma 4. *Let (M, d) be a metric space, $K > 0$ and let $(x_n, y_n)_{n \in \mathbb{N}}$ be a sequence of pairs of points from M satisfying the following three conditions.*

- (i) *For every $n \in \mathbb{N}$, we have $x_n \neq y_n$.*
- (ii) *For every $n, m \in \mathbb{N}$, we have $x_m \notin U(y_n, K \cdot d(y_n, x_n))$.*
- (iii) *For every $n \neq m$, we have $U(y_n, K \cdot d(y_n, x_n)) \cap U(y_m, K \cdot d(y_m, x_m)) = \emptyset$.*

Then $\ell_\infty \hookrightarrow \text{Lip}_0(M)$.

Proof. We may without loss of generality assume that $0 = x_1$ (because $\text{Lip}_0(M) \ni f \mapsto f - f(x_1)$ is a linear isometry onto the space of Lipschitz functions g with $g(x_1) = 0$). For every $n \in \mathbb{N}$ we define $f_n(x) := \max\{d(y_n, x_n) - \frac{d(y_n, x)}{K}, 0\}$, $x \in M$. Then $f_n \in \text{Lip}(M)$. Moreover, it is easy to see that $\|f_n\|_{\text{Lip}} \leq \frac{1}{K}$ and $\|f\|_{\infty} = d(y_n, x_n)$. By (ii), we have $K \cdot d(x_n, y_n) \leq d(x_1, y_n)$; hence, $f_n(0) = 0$.

Notice that condition (iii) implies that if $f_n(x) \neq 0$, then for every $m \neq n$ we have $f_m(x) = 0$. For every $x \in M$, we denote by $n(x)$ the unique $n \in \mathbb{N}$ with $f_n(x) \neq 0$ if it exists; otherwise, we put $n(x) := 1$. Finally, we define $T : \ell_{\infty} \rightarrow \text{Lip}_0(M)$ by

$$T(\alpha)(x) := \alpha(n(x)) \cdot f_{n(x)}(x), \quad \alpha = (\alpha(n))_{n \in \mathbb{N}} \in \ell_{\infty}.$$

First, we will show that T is linear and $\|T\| \leq \frac{2}{K}$. It is easy to see that T is linear; hence, it suffices to show that for $\alpha = (\alpha(n))_{n \in \mathbb{N}} \in \ell_{\infty}$ with $\|\alpha\| = 1$, we have $\|T(\alpha)\|_{\text{Lip}} \leq \frac{2}{K}$. Fix $x, y \in M$. We need to show that $|T(\alpha)(x) - T(\alpha)(y)| \leq \frac{2}{K}d(x, y)$. If $n(x) = n(y)$, this is easy because $f_{n(x)}$ is $\frac{1}{K}$ -Lipschitz. Hence, we may assume that $n(y) \neq n(x)$. Thus, we have $f_{n(x)}(y) = 0 = f_{n(y)}(x)$ and $|T(\alpha)(x) - T(\alpha)(y)| \leq f_{n(x)}(x) + f_{n(y)}(y) = |f_{n(x)}(x) - f_{n(x)}(y)| + |f_{n(y)}(y) - f_{n(y)}(x)| \leq \frac{2}{K}d(x, y)$.

In order to see that T is an isomorphism, we will use condition (ii). Fix $\alpha = (\alpha(n))_{n \in \mathbb{N}} \in \ell_{\infty}$ and $N \in \mathbb{N}$. By (ii), for every $k \in \mathbb{N}$, we have $f_k(x_N) = 0$; hence $T(\alpha)(x_N) = 0$ and we have

$$|T(\alpha)(x_N) - T(\alpha)(y_N)| = |T(\alpha)(y_N)| = |\alpha(N)|f_N(y_N) = |\alpha(N)|d(x_N, y_N).$$

Therefore, $\|T(\alpha)\|_{\text{Lip}} \geq |\alpha(N)|$ and, since N was arbitrary, $\|T(\alpha)\|_{\text{Lip}} \geq \|\alpha\|_{\infty}$. \square

The following result is the main step towards the proof of Theorem 1.

Theorem 5. *Let M be an infinite metric space. Then $\ell_{\infty} \hookrightarrow \text{Lip}_0(M)$.*

Proof. First, note that we may without generality assume that M is complete, because otherwise we take the completion N of M and use the obvious fact that $\text{Lip}_0(\overline{A})$ is linearly isometric to $\text{Lip}_0(A)$ for every $A \subset N$; in particular, $\text{Lip}_0(N)$ is linearly isometric to $\text{Lip}_0(M)$.

Now, we will prove the statement considering several cases. In each of them we will find a sequence of pairs of points from M satisfying the assumptions of Lemma 4.

Case 1. *M is unbounded; i.e., for every $K > 0$ there are $x, y \in M$ with $d(x, y) > K$.*

Proof for Case 1. Pick a sequence $(z_n)_{n=1}^{\infty}$ in M such that, for every $n \in \mathbb{N}$ we have $d(z_{n+1}, 0) > 2d(z_n, 0)$. Now, for each $n \in \mathbb{N}$, put $x_n := z_{2n-1}$ and $y_n := z_{2n}$. We will show that the sequence $(x_n, y_n)_{n \in \mathbb{N}}$ satisfies the assumptions of Lemma 4 with $K = \frac{1}{3}$.

Obviously, (i) is satisfied. Further, for $n < m$ we have

$$d(z_n, z_m) \leq d(z_n, 0) + d(z_m, 0) < (1 + 2^{-(m-n)})d(z_m, 0), \quad (1)$$

$$d(z_n, z_m) \geq d(z_m, 0) - d(z_n, 0) > (1 - 2^{-(m-n)})d(z_m, 0). \quad (2)$$

Let us show that (ii) holds. We need to show that, for $n, m \in \mathbb{N}$, we have

$$d(z_{2m-1}, z_{2n}) \geq \frac{1}{3} \cdot d(z_{2n-1}, z_{2n}). \quad (3)$$

This is obvious if $m = n$. If $m < n$, then we have

$$d(z_{2m-1}, z_{2n}) \stackrel{(2)}{>} (1 - 2^{-(2n-2m+1)})d(z_{2n}, 0) \stackrel{(1)}{>} (1 - 2^{-3})(1 + 2^{-1})^{-1}d(z_{2n}, z_{2n-1}).$$

If $n < m$, then we have

$$d(z_{2m-1}, z_{2n}) \stackrel{(2)}{>} (1 - 2^{-1})d(z_{2m-1}, 0) > (1 - 2^{-1})d(z_{2n}, 0) \stackrel{(1)}{>} (1 - 2^{-1})(1 + 2^{-1})^{-1}d(z_{2n}, z_{2n-1}).$$

This proves (3); hence, condition (ii) from Lemma 4 is satisfied.

Finally, in order to see that (iii) holds, it is sufficient to see that, for $n \neq m$, we have

$$d(z_{2m}, z_{2n}) > \frac{1}{3} \cdot (d(z_{2n-1}, z_{2n}) + d(z_{2m-1}, z_{2m})). \quad (4)$$

We may assume that $n < m$ and then we have

$$\begin{aligned} d(z_{2n-1}, z_{2n}) + d(z_{2m-1}, z_{2m}) &\stackrel{(1)}{<} (1 + 2^{-1})^2 d(z_{2m}, 0) \stackrel{(2)}{<} (1 + 2^{-1})^2 (1 - 2^{-(2m-2n)})^{-1} d(z_{2m}, z_{2n}) \\ &\leq (1 + 2^{-1})^2 (1 - 2^{-2})^{-1} d(z_{2m}, z_{2n}) = 3d(z_{2m}, z_{2n}). \end{aligned}$$

This proves (4); hence, condition (iii) from Lemma 4 is satisfied. \square

Case 2. *M is bounded and there is a closed infinite subset $N \subset M$ such that each point $n \in N$ is isolated in N .*

Proof for Case 2. Fix N as above. Since M is bounded, there is $D > 0$ such that, for every $x, y \in M$, we have $d(x, y) \leq D$. Since N does not contain any nontrivial Cauchy sequence, there is an infinite subset $P \subset N$ which is uniformly discrete; i.e., there is $C > 0$ such that, for every $x, y \in P$ with $x \neq y$, we have $d(x, y) \geq C$ (this is a simple exercise using the classical Ramsey theorem, see e.g. [24, Exercise 5.5]). Fix a one-to-one sequence $(a_n)_{n=0}^\infty$ of points from P and, for every $n \in \mathbb{N}$, put $y_n = a_n$ and $x_n = a_0$. It remains to verify that the sequence $(x_n, y_n)_{n \in \mathbb{N}}$ satisfies the assumptions of Lemma 4 with $K = \min\{\frac{C}{2D}, 1\}$. It is clear that conditions (i) and (ii) are satisfied, because $K \leq 1$. Moreover, for every $n \in \mathbb{N}$, we have $K \cdot d(x_n, y_n) \leq KD \leq C/2$; hence, $U(y_n, K \cdot d(x_n, y_n)) \subset U(y_n, C/2)$ and, since $(y_n)_{n \in \mathbb{N}}$ is C -discrete, the balls are pairwise disjoint. This verifies condition (iii) from Lemma 4. \square

Case 3. *M is bounded and it contains infinitely many limit points.*

Proof for Case 3. First, let us assume there is a sequence $(y_n)_{n \in \mathbb{N}}$ consisting of limit points in M with $y_n \rightarrow y$. Then, for each $n \in \mathbb{N}$, put $r_n := \text{dist}(y_n, \{y_m : m \neq n\}) > 0$ and pick some $x_n \in U(y_n, r_n/2)$ with $x_n \neq y_n$. Then it is easy to see that the sequence $(x_n, y_n)_{n \in \mathbb{N}}$ satisfies the assumptions of Lemma 4 with $K = 1$.

Otherwise, the set N consisting of all the limit points in M satisfies the assumptions of Case 2. \square

Note that now it remains to handle the case when M is compact and it contains a nontrivial convergent sequence consisting of isolated points. Indeed, by the already proven Cases 1-3, we may assume M is bounded and contains only finitely many limit points. Then either M is compact, or there is an infinite closed set of isolated points in M and we may apply Case 2.

Case 4. *M is compact and it contains a nontrivial convergent sequence consisting of isolated points.*

Proof for Case 4. Let $(a_n)_{n \in \mathbb{N}}$ be a nontrivial convergent sequence consisting of isolated points with the limit point a . It is easy to construct by induction a sequence $(x_n, y_n)_{n \in \mathbb{N}}$ of pairs of points from M with

- (a) $\forall n \in \mathbb{N} : y_n \in \{a_n : n \in \mathbb{N}\}$ and $y_n \notin \{y_m : m < n\} \cup \{x_m : m < n\}$,
- (b) $\forall n \in \mathbb{N} : d(y_n, a) < \min\{d(y_n, y_m) : m < n\}$ and
- (c) for every $n \in \mathbb{N}$, we pick x_n to be any point with $d(x_n, y_n) = \text{dist}(y_n, M \setminus \{y_n\})$.

Now, having such a sequence $(x_n, y_n)_{n \in \mathbb{N}}$, it remains to check that it satisfies the assumptions of Lemma 4 with $K = 1$. Obviously, (i) is satisfied. Moreover, we have $d(x_n, y_n) \leq d(y_n, x)$ for every $x \in M \setminus \{y_n\}$ and so in order to verify (ii), it is enough to observe that, for $n, m \in \mathbb{N}$, we have $x_n \neq y_m$. This follows from (a) for $n < m$, from (c) for $n = m$ and from (b) for $n > m$, because in the last case we have $d(x_n, y_n) \leq d(y_n, a) < d(y_n, y_m)$.

It remains to verify (iii). But this is easy, because, for every $n \in \mathbb{N}$, by the choice of x_n we have $U(y_n, d(x_n, y_n)) = \{y_n\}$. \square

Since the cases mentioned above cover all the possibilities, this completes the proof of Theorem 5. \square

Proof of Theorem 1. This is a consequence of Theorem 5. Indeed, it is a classical result that, for every Banach space X , $\ell_\infty \hookrightarrow X^*$ if and only if ℓ_1 is isomorphic to a complemented subspace of X [4, Theorem 4]; hence, (i) follows. Since any complemented subspace of a $\mathcal{C}(K)$ space contains c_0 , see e.g. [25, Theorem 5.1], from (i) we get (ii) because c_0 is not isomorphic to a subspace of ℓ_1 . Since the dual space contains c_0 , it is not weakly sequentially complete and so it is not isomorphic to $L^1(\mu)$ [28, Corollary III.C.14]. Therefore, X is not isomorphic to any L^1 -predual; in particular, not to the Gurarii space [15]; see also [12, Theorem 2.17]. As it is well known that ℓ_1 is projectively universal, i.e., for any separable Banach space Y there exists a bounded linear operator from ℓ_1 onto Y , the same is true for X since ℓ_1 is complemented there. \square

Remark 6. From the assumptions of Lemma 4 it is possible not only to deduce that ℓ_1 is isomorphic to a complemented subspace of $\mathcal{F}(M)$, but it is even possible to describe relatively easily this subspace. Let us assume that $(x_n, y_n)_{n \in \mathbb{N}}$ is as in Lemma 4. For each $n \in \mathbb{N}$, put $e_n := \frac{\delta_{y_n} - \delta_{x_n}}{d(y_n, x_n)} \in \mathcal{F}(M)$. Then, using similar proof as in Lemma 4 we get that $(e_n)_{n \in \mathbb{N}}$ is $2/K$ -equivalent to the ℓ_1 basis. Moreover, consider functions $(f_n)_{n \in \mathbb{N}}$ from the proof of Lemma 4 and define $r : M \rightarrow \mathcal{F}(M)$ by $r(x) := \sum_{n \in \mathbb{N}} f_n(x) e_n$, $x \in M$. Then it is possible to verify that r is a $2/K$ -Lipschitz. Using the universal property of r we find $P : \mathcal{F}(M) \rightarrow \mathcal{F}(M)$ with $P \circ \delta = r$ and $\|P\| \leq 2/K$. Finally, one can verify that P is actually a projection onto $\overline{\text{span}}\{e_n : n \in \mathbb{N}\}$.

3. EMBEDDING INTO L_1

The purpose of this section is to prove Theorem 2. In order to prove it, we will need the following result. The proof is just a modification of the arguments from [19].

Theorem 7. *For any measure μ , $\mathcal{F}([0, 1]^2) \not\hookrightarrow L_1(\mu)$.*

Proof. In order to shorten our notation, put $I := [0, 1]^2$. If there is a measure μ with $\mathcal{F}(I) \hookrightarrow L_1(\mu)$, then there is a continuous linear mapping from $L_\infty(\mu)$ onto $\text{Lip}_0(I)$. Since $L_\infty(\mu)$ is a commutative C^* -algebra, there exists a compact Hausdorff space K such that $L_\infty(\mu)$ is isometric to $\mathcal{C}(K)$. Hence, it suffices to show that there does not exist a bounded linear mapping $T : \mathcal{C}(K) \rightarrow \text{Lip}_0(I)$ which is onto. We only show that the “identity” mapping $id : \text{Lip}_0(I) \rightarrow W^{1,1}(I)$ is absolutely summing. Then the rest can be proved just copying line by line the arguments from [19, Theorem 3], where this statement is proved for the space $\mathcal{C}^1(I)$ instead of $\text{Lip}_0(I)$ using the fact that “identity” mapping $id : \mathcal{C}^1(I) \rightarrow W^{1,1}(I)$ is “absolutely summing” ($W^{1,1}(I)$ is the Sobolev space). So consider the “identity” mapping $id : \text{Lip}_0(I) \rightarrow W^{1,1}(I)$. More precisely, having a Lipschitz function f , we denote by $[f]$ the equivalence class containing all the functions which are equal to f almost everywhere. The “identity” mapping is the mapping $f \mapsto [f]$. By the classical Rademacher’s theorem, see e.g. [23], every Lipschitz function defined on I is almost everywhere differentiable and so it is possible to put $\|[f]\|_W := \int_{[0,1]^2} (|f(x, y)| + |\partial_1 f(x, y)| + |\partial_2 f(x, y)|) dx dy$. It is immediate that $\|[f]\|_W \leq 3\|f\|_{\text{Lip}}$ and it remains to show that the mapping $f \mapsto [f]$ is absolutely summing; i.e., there is a constant C such that whenever $(f_i)_{i=1}^m$ are functions from $\text{Lip}_0(I)$, then

$$\sum_{i=1}^m \|[f_i]\|_W \leq C \sup \left\{ \sum_{i=1}^m |x^*(f_i)| : x^* \in \text{Lip}_0(I)^*, \|x^*\| \leq 1 \right\}.$$

Let us define $\Phi : \text{Lip}_0([0, 1]^2) \rightarrow L_\infty(I) \oplus_1 L_\infty(I) \oplus_1 L_\infty(I)$ by $\text{Lip}_0(I) \ni f \mapsto \Phi(f) := (f, \partial_1 f, \partial_2 f)$. Note that Φ is a linear bounded operator. Further, consider $\Psi : L_\infty(I) \oplus_1 L_\infty(I) \oplus_1 L_\infty(I) \rightarrow L_1(I) \oplus_1 L_1(I) \oplus_1 L_1(I)$ defined as the identity. It is a standard fact, see e.g. [2, Remark 8.2.9], that the identity operator from $L_\infty(I)$ to $L_1(I)$ is absolutely summing; hence, Ψ is absolutely summing.

It is a classical fact that composition of a bounded operator with an absolutely summing one is absolutely summing, see e.g. [2, Proposition 8.2.5]. Hence, $id = \Psi \circ \Phi$ is absolutely summing. \square

Remark 8. The result that $\mathcal{F}(\mathbb{R}^2) \not\hookrightarrow L_1$ is often mentioned as a result of A. Naor and G. Schechtmann [22]. The proof above shows that, using minor modifications, it actually follows already from [19].

The rest of this section is devoted to the proof of Theorem 2. First, we construct the countable compact space with one accumulation point and then in a series of claims we prove the statement.

For every $n \geq 2$, let (A_n, d_n) be the set $\{(\frac{i}{n^2}, \frac{j}{n^2}) : 0 \leq i, j \leq n\}$ equipped with the Euclidean distance d_n inherited from \mathbb{R}^2 . Denote by K the amalgamated metric sum of A_n 's over 0. That is, we take K to be the disjoint union $\coprod_n A_n$ with the zero element $(0, 0)$ identified in all of them. The metric d on K is defined as follows. For $a, b \in K$ we set

$$d(a, b) = \begin{cases} d_n(a, b) & \exists n(a, b \in A_n), \\ d_n(a, 0) + d_m(b, 0) & a \in A_n, b \in A_m, n \neq m. \end{cases}$$

It is easy to check that K is a countable compact metric space; in fact, it is a convergent sequence, i.e., it has only one accumulation point - the zero.

Claim 1. $\mathcal{F}(K)$ is isometric to $\bigoplus_{\ell_1} \mathcal{F}(A_n)$.

Proof. This is easy and proved e.g. in [17, Proposition 5.1]. \square

For every n , consider the set $nA_n := \{(\frac{i}{n}, \frac{j}{n}) : 0 \leq i, j \leq n\}$ again equipped with the Euclidean distance. Clearly, $\mathcal{F}(nA_n)$ is isometric to $\mathcal{F}(A_n)$. Indeed, since both spaces are finite-dimensional, it suffices to find an isometry of their duals; the mapping $\phi : \text{Lip}_0(A_n) \rightarrow \text{Lip}_0(nA_n)$ defined by $\phi(f)(x) := nf(\frac{x}{n})$, $x \in nA_n$, $f \in \text{Lip}_0(A_n)$ is such an isometry. As a consequence we get the following.

Claim 2. $\mathcal{F}(K)$ is linearly isometric to $\bigoplus_{\ell_1} \mathcal{F}(nA_n)$.

Since nA_n , for each n , is a subset of $[0, 1]^2$ we may and will consider $\mathcal{F}(nA_n)$ as a subspace of $\mathcal{F}([0, 1]^2)$. Notice that $\bigcup_n nA_n$ is dense in $[0, 1]^2$. We need one more technical claim which says that finite dimensional subspaces of $\mathcal{F}([0, 1]^2)$ can be approximated by finite dimensional subspaces of $\mathcal{F}(nA_n)$ for large enough n . In the following, by d_{BM} we denote the Banach-Mazur distance.

Claim 3. Let $E \subseteq \mathcal{F}([0, 1]^2)$ be a finite dimensional subspace and let $\varepsilon > 0$ be arbitrary. Then there exist $n \in \mathbb{N}$ and a finite dimensional subspace $E' \subseteq \mathcal{F}(nA_n)$ such that $d_{BM}(E, E') < 1 + \varepsilon$.

Proof. Let e_1, \dots, e_m be a basis of E . Since all norms on E are equivalent, there is $D > 0$ such that for all $a \in \mathbb{R}^m$ we have $\sum_{i=1}^m |a(i)| \leq D \|\sum_{i=1}^m a(i)e_i\|$. Fix $\delta = \frac{\varepsilon}{2+\varepsilon}$, i.e., such that $\frac{1+\delta}{1-\delta} = 1 + \varepsilon$. Each e_i can be $(\delta/2mD)$ -approximated by some linear combination of elements from $\text{span}\{\delta_y : y \in [0, 1]^2\}$. Without loss of generality, we may assume that for each $i \leq m$ such a linear combination is of the same length. So for each $i \leq m$, we choose some $\alpha_1^i \delta_{x_1^i} + \dots + \alpha_l^i \delta_{x_l^i} \in \text{span}\{\delta_y : y \in [0, 1]^2\}$ such that $\|e_i - (\alpha_1^i \delta_{x_1^i} + \dots + \alpha_l^i \delta_{x_l^i})\| < \delta/2mlD$.

Now, since $\bigcup_{n \in \mathbb{N}} nA_n$ is dense in $[0, 1]^2$, if we take n large enough then for every $i \leq m$ and $j \leq l$ we can find $a_j^i \in nA_n$ such that $\|\delta_{x_j^i} - \delta_{a_j^i}\| < \frac{\delta}{2ml\alpha D}$, where $\alpha = \max\{|\alpha_j^i| : j \leq l, i \leq m\}$. Consequently, for every $i \leq m$ we get

$$\|\alpha_1^i \delta_{x_1^i} + \dots + \alpha_l^i \delta_{x_l^i} - (\alpha_1^i \delta_{a_1^i} + \dots + \alpha_l^i \delta_{a_l^i})\| < \alpha \cdot l \cdot \frac{\delta}{2ml\alpha D} = \delta/2mD.$$

Thus, if for every $i \leq m$ we denote $\alpha_1^i \delta_{a_1^i} + \dots + \alpha_i^i \delta_{a_i^i}$ by e'_i , we have $\|e_i - e'_i\| < \delta/mD$. Hence, for any $a \in \mathbb{R}^m$, we have $\|\sum_{i=1}^m a(i)e_i - \sum_{i=1}^m a(i)e'_i\| < \delta/D(\sum_{i=1}^m |a(i)|) \leq \delta\|\sum_{i=1}^m a(i)e_i\|$ and, consequently,

$$\left\| \sum_{i=1}^m a(i)e'_i \right\| < (1 + \delta) \left\| \sum_{i=1}^m a(i)e_i \right\| \quad \text{and} \quad \left\| \sum_{i=1}^m a(i)e_i \right\| < \left\| \sum_{i=1}^m a(i)e'_i \right\| + \delta \left\| \sum_{i=1}^m a(i)e_i \right\|.$$

Denote by E' the subspace $\text{span}\{e'_i : i \leq m\} \subseteq \mathcal{F}(nA_n)$. Using the above, the linear mapping determined by sending e_i to e'_i , for $i \leq m$, is a witness of the fact that $d_{BM}(E, E') < \frac{1+\delta}{1-\delta} = 1+\varepsilon$. \square

Let us now formulate a result of Lindenstrauss and Pełczyński that will help us finish the proof.

Theorem 9 (Theorem 7.1 in [21]). *Let X be a Banach space and fix $\lambda \geq 1$. If for every finite dimensional subspace E of X there exists a finite dimensional subspace E' of ℓ_1 such that $d_{BM}(E, E') \leq \lambda$, then there exists a measure μ and a subspace Y of $L_1(\mu)$ such that $d_{BM}(X, Y) \leq \lambda$.*

We are now ready to finish the proof of Theorem 2. By Theorem 7 we have that $\mathcal{F}([0, 1]^2)$ does not embed into $L_1(\mu)$ for any measure μ . However, then by Theorem 9 we get that for every $N \in \mathbb{N}$ there exists a finite dimensional subspace E_N of $\mathcal{F}([0, 1]^2)$ such that for every finite dimensional subspace F of ℓ_1 we have $d_{BM}(E_N, F) > N$. Using Claim 3, for each N we can find some $n(N) \in \mathbb{N}$ and finite dimensional subspace $E_{n(N)}$ of $\mathcal{F}(n(N)A_{n(N)})$ such that $d_{BM}(E_N, E_{n(N)}) < 2$.

Assume now that $\mathcal{F}(K)$ embeds into L_1 via some linear embedding of norm less than $N/8$ for some $N \in \mathbb{N}$. By Claim 2, $\oplus_{\ell_1} \mathcal{F}(nA_n)$ embeds into L_1 via some linear embedding T of norm less than $N/8$. Now, T restricted on $E_{n(N)} \subseteq \mathcal{F}(n(N)A_{n(N)})$ has still norm bounded by $N/8$. In particular, there is some finite dimensional subspace $Y_{n(N)}$ of L_1 such that $d_{BM}(E_{n(N)}, Y_{n(N)}) \leq N/8$. Since L_1 is finitely representable in ℓ_1 , see [2, Proposition 11.1.7], there exists a finite dimensional subspace Y_N of ℓ_1 such that $d_{BM}(Y_N, Y_{n(N)}) < 2$.

Now putting all these inequalities together we get

$$\frac{N}{8} \geq d_{BM}(E_{n(N)}, Y_{n(N)}) \geq \frac{d_{BM}(E_N, Y_N)}{d_{BM}(E_N, E_{n(N)}) \cdot d_{BM}(Y_N, Y_{n(N)})} > \frac{N}{4}$$

and that is a contradiction finishing the proof.

Remark 10. The paper [17] was published (see [18]). However, in the published version the statement [17, Proposition 5.1], which we cite in the proof of Claim 1 above is missing. Thus, we would like to sketch the easy proof of it here.

Consider

$$\Phi : \bigoplus_{\ell_\infty} \text{Lip}_0(A_n) \rightarrow \text{Lip}_0(K)$$

defined by $\Phi((f_n))(x) = f_n(x)$, $(f_n) \in \bigoplus_{\ell_\infty} \text{Lip}_0(A_n)$, $x \in A_n$. Then it is easy to verify that Φ is an isometry onto and $w^* - w^*$ homeomorphism. Hence, it is the adjoint of an isometry from $\mathcal{F}(K)$ onto $\bigoplus_{\ell_1} \mathcal{F}(A_n)$.

Remark 11. It has been observed by G. Lancien and A. Procházka that our method of proof actually gives that K from the statement of Theorem 2 can be taken as a subset of $[0, 1]^2$ and that there does not exist a bi-Lipschitz embedding of $\mathcal{F}(K)$ into L_1 . Let us sketch the argument here.

First, the only place where we used the metric of K was to prove Claim 1. However, it is easy to see that taking a sequence $(k_n)_{n \in \mathbb{N}}$ increasing fast enough, we have that $\mathcal{F}(\bigcup_{n \in \mathbb{N}} A_{k_n})$ is linearly isomorphic to $\bigoplus_{\ell_1} \mathcal{F}(A_{k_n})$ (using the same mapping Φ as in Remark 10), which would be enough for the rest of the proof. Hence, we may have $K = \bigcup_{n \in \mathbb{N}} A_{k_n}$. Moreover, our proof gives that $\mathcal{F}(K)$ does not linearly embed into any Banach space finitely representable in ℓ_1 . If there was a

bi-Lipschitz embedding of $\mathcal{F}(K)$ into L_1 then, by [3, Corollary 7.10], $\mathcal{F}(K)$ embeds linearly into $(L_1)^{**}$ which is by the principle of local reflexivity [2, Theorem 11.2.4] finitely represented in L_1 (in particular, $(L_1)^{**}$ is finitely represented in ℓ_1 because L_1 is), a contradiction.

4. EMBEDDING OF c_0

Let M be a separable metric space that contains a bi-Lipschitz copy of every separable metric space. By [14, Theorems 2.12 and 3.1], we have $X \hookrightarrow \mathcal{F}(X)$ for every separable Banach space X ; therefore, $\mathcal{F}(M)$ is a universal separable Banach space, i.e., $\mathcal{F}(M)$ contains an isomorphic copy of every separable Banach space. Note that by the result of Aharoni [1] this is equivalent to the condition that M contains a bi-Lipschitz copy of c_0 . Y. Dutrieux and V. Ferenczi in [9] asked for the converse, see Question 1. In this section we prove Theorem 3, making a partial progress towards the answer to this question.

Let M be either $[0, 1]^n$ or \mathbb{R}^n . By $\mathcal{C}^1(M)$ we denote the space of functions $F : M \rightarrow \mathbb{R}$ whose derivatives of order ≤ 1 are continuous on M . For $F \in \mathcal{C}^1(M)$ we define $\|F\|_\infty^1 := \max\{\|F\|_\infty, \|\partial_{x_i} F\|_\infty : i \leq n\}$. It is well-known that the space $(\mathcal{C}^1([0, 1]^n), \|\cdot\|_\infty^1)$ is a Banach space.

The following result was essentially proved by J. Bourgain [5], [6]. The result of J. Bourgain concerns the space of smooth functions over n -dimensional torus; however, the same proof works for the n -dimensional cube. We refer also to [28], where a more detailed proof of the result of J. Bourgain may be found (use Example III.D.30 and Theorem III.D.31 and conclude similarly as in the proof of Corollary III.C.14).

Theorem 12. *For every $n \in \mathbb{N}$, the Banach space $(\mathcal{C}^1([0, 1]^n))^*$ is weakly sequentially complete, i.e., weakly Cauchy sequences are weakly convergent.*

Lemma 13. *Let $A \subset \mathbb{R}^n$ be a finite set and $f : A \rightarrow \mathbb{R}$ a 1-Lipschitz function (on \mathbb{R}^n we consider Euclidean norm). Then, for every $\varepsilon > 0$, there exists $g \in \mathcal{C}^1(\mathbb{R}^n)$, an extension of f (i.e., $g \supset f$), with $\|g\|_\infty^1 < \max\{\|f\|_\infty, 1\} + \varepsilon$.*

Proof. Find $\delta > 0$ such that the balls $\{B(a, 2\delta) : a \in A\}$ are pairwise disjoint. Fix some even Lipschitz function $\tau \in \mathcal{C}^1(\mathbb{R})$ with $\tau(0) = 1$, $\|\tau\|_\infty \leq 1$ and $\{x : \tau(x) \neq 0\} \subset (-\delta, \delta)$; e.g.

$$\tau(x) = \begin{cases} e^{-\frac{1}{\delta^2 - x^2} + \frac{1}{\delta^2}} & |x| < \delta, \\ 0 & \text{otherwise.} \end{cases}$$

Let K be such that τ is K -Lipschitz and $K > 1$.

We may assume that $0 \in A$ and $f(0) = 0$. First, we extend f to a 1-Lipschitz function defined on \mathbb{R}^n ; see e.g. [16, Lemma 7.39]. We call this extension again f . Now, we find a 1-Lipschitz $\tilde{g} \in \mathcal{C}^1(\mathbb{R}^n)$ with $\|f - \tilde{g}\|_\infty < \varepsilon/2K$; e.g. using the standard integral convolution [16, Lemma 7.1].

For $a \in A$ define $\phi_a : \mathbb{R}^n \rightarrow \mathbb{R}$ by $\phi_a(x) = (f(a) - \tilde{g}(a))\tau(\|x - a\|)$, $x \in \mathbb{R}^n$. Then $h := \sum_{a \in A} \phi_a$ is a well-defined $\varepsilon/2$ -Lipschitz function such that $\|h\|_\infty \leq \varepsilon/2K$ and $h(a) = f(a) - \tilde{g}(a)$ for every $a \in A$. Moreover, since on a Hilbert space the norm is smooth everywhere except 0 and since the function τ is even, it is easy to observe that $h \in \mathcal{C}^1(\mathbb{R}^n)$. It remains to put $g := \tilde{g} + h$. Then we have $g \in \mathcal{C}^1(\mathbb{R}^n)$, $\|g\|_\infty < \|f\|_\infty + \varepsilon$ and g is $(1 + \varepsilon/2)$ -Lipschitz; hence, $\|g\|_\infty^1 < \max\{\|f\|_\infty, 1\} + \varepsilon$. \square

Remark 14. Note that it follows from Lemma 13 that whenever A is a finite set in $[0, 1]^n$ and f a 1-Lipschitz function on A with $f(0) = 0$, there is $g \in \mathcal{C}^1(\mathbb{R}^n)$ with $\|g\|_\infty^1 \leq \sqrt{n} + 1$. Therefore, by [10, Theorem 1], there is a linear extension operator $T : \text{Lip}_0(A) \rightarrow \text{Lip}_0(\mathbb{R}^n)$ with norm depending only on n ; hence, $T^*|_{\mathcal{F}(\mathbb{R}^n)}$ is a projection from $\mathcal{F}(\mathbb{R}^n)$ onto $\mathcal{F}(A)$. Consequently, whenever we have $M \subset [0, 1]^n$ and $A \subset M$ a finite set, $\mathcal{F}(A)$ is $C(n)$ -complemented in $\mathcal{F}(M)$, where the constant $C(n)$ depends only on the dimension n . This gives another proof of the fact that $\mathcal{F}(M)$ has BAP whenever $M \subset [0, 1]^n$ [20, Proposition 2.3].

Lemma 15. *For every $n \in \mathbb{N}$, there is an isomorphism of $\mathcal{F}([0, 1]^n)$ into $(\mathcal{C}^1([0, 1]^n))^*$.*

Proof. Put $Y = \{f \in \mathcal{C}^1([0, 1]^n) : f(0) = 0\}$. Then Y is a closed subspace of codimension 1; hence, it is complemented and Y^* is isomorphic to a subspace of $(\mathcal{C}^1([0, 1]^n))^*$. For every $x \in [0, 1]^n$ we define $T(\delta_x) \in Y^*$ by $T(\delta_x)(f) := f(x)$, $f \in Y$. Extend T linearly to the set $\text{span}\{\delta_x : x \in [0, 1]^n\}$. Now, it is enough to verify that T is an isomorphism into Y^* .

Fix an element $\mu \in \text{span}\{\delta_x : x \in [0, 1]^n\}$. There are $k \in \mathbb{N}$, $\alpha \in \mathbb{R}^k$ and $x_1, \dots, x_k \in [0, 1]^n$ with $\mu = \sum_{i=1}^k \alpha(i) \delta_{x_i}$. We have to find constants $C > 0$ and $D > 0$ with

$$\begin{aligned} C \sup \left\{ \left| \sum_{i=1}^k \alpha(i) f(x_i) \right| : \|f\|_\infty^1 \leq 1, f(0) = 0 \right\} &\leq \sup \left\{ \left| \sum_{i=1}^k \alpha(i) f(x_i) \right| : \|f\|_{\text{Lip}} \leq 1, f(0) = 0 \right\} \\ &\leq D \sup \left\{ \left| \sum_{i=1}^k \alpha(i) f(x_i) \right| : \|f\|_\infty^1 \leq 1, f(0) = 0 \right\}. \end{aligned}$$

The existence of constant C follows from the basic fact that every function with total differential bounded by K is K -Lipschitz; see [26, Theorem 9.19]; hence, we may put $C = 1/\sqrt{n}$. The existence of D follows from Lemma 13, which gives $D = \sqrt{n}$. \square

Proof of Theorem 3. By [18, Corollary 3.3], $\mathcal{F}(\mathbb{R}^n)$ is isomorphic to $\mathcal{F}([0, 1]^n)$. Hence, $\mathcal{F}(\mathbb{R}^n)$ is weakly sequentially complete by Theorem 12 and Lemma 15. Finally, using the fact mentioned in Section 1 that $\mathcal{F}(M)$ is isometric to a subspace of $\mathcal{F}(\mathbb{R}^n)$, we see that $\mathcal{F}(M)$ is weakly sequentially complete. Consequently, c_0 does not embed isomorphically into $\mathcal{F}(M)$ because, as it is well known and easy to prove, c_0 is not weakly sequentially complete. \square

5. OPEN PROBLEMS

As it was mentioned in Section 4, if M contains a bi-Lipschitz copy of c_0 , then $\mathcal{F}(M)$ is a universal separable Banach space. Hence, we have quite a rich family of universal separable Banach spaces. By Theorem 1, they are all different from $\mathcal{C}(K)$ spaces and from the Gurarii space. One example is Pełczyński's universal basis space \mathbb{P} (which is unique up to isomorphism). This space is isomorphic to $\mathcal{F}(\mathbb{P})$, see [14, p. 139]. Another example is the Holmes space, i.e., the Lipschitz-free space over the Urysohn universal metric space. By [11, Theorem 4.2], the Holmes space is not isomorphic to \mathbb{P} . By [9, Theorem 5], $\mathcal{F}(c_0)$ is isomorphic to each $\mathcal{F}(\mathcal{C}(K))$. It could be of some interest to find out what isomorphic types of universal Banach spaces we are able to get using the Lipschitz-free construction. For example, the following seems to be open.

Question 2. *Is $\mathcal{F}(c_0)$ isomorphic to the Holmes space or to \mathbb{P} ?*

In the light of Theorem 3 it is also natural to ask the following.

Question 3. *Is it true that $c_0 \hookrightarrow \mathcal{F}(\ell_2)$?*

Note that c_0 does not bi-Lipschitz embed into ℓ_p ($1 \leq p < \infty$), see e.g. [3, p. 169]. Hence, the negative answer to the above question would be a partial progress towards the answer to Question 1. Similarly, we do not know the answer to the following question.

Question 4. *Is it true that $c_0 \hookrightarrow \mathcal{F}(\ell_1)$?*

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